LIEB-THIRRING INEQUALITY FOR A MODEL OF PARTICLES WITH POINT INTERACTIONS

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Abstract. We consider a model of quantum-mechanical particles interacting via point interactions of infinite scattering length. In the case of fermions we prove a Lieb-Thirring inequality for the energy, i.e., we show that the energy is bounded from below by a constant times the integral of the particle density to the power $\frac{5}{3}$.

To Elliott Lieb, in appreciation of many years of fruitful and inspiring collaboration.

1. Introduction

Quantum-mechanical models of particles with point interactions have become relevant in cold-atom physics, where one encounters systems where the range of the interactions among the atoms can be much shorter than their average distance, while the scattering length can be much larger. In the limit of zero interaction range and infinite scattering length, which is referred to as the "unitary limit", one is left with a system without intrinsic length scale. For further discussion of this topic we refer to [14, 3, 4, 23, 11] and, in particular, to the articles in [24].

While the two-body problem for particles interacting via point interactions is well understood [1], it remains an open problem to establish the existence of a model for $N > 2$ particles with only two-body point interactions (see [6, 10] and references there). Such a model can only be expected to exist for spin $\frac{1}{2}$ fermions, due to the Efimov effect [9], i.e., the existence of three-body bound states despite the absence of two-body bound states. We consider here a model with more complicated point interactions which is manifestly well-defined, in the sense the quadratic form for the energy is positive. This model is well-defined even for bosons, and does not allow for any bound states even in this case.

We shall prove that the model we consider satisfies a Lieb-Thirring inequality, i.e., the energy of fermions is bounded from below a semiclassical approximation to the kinetic energy. Up to the value of the constant, this inequality is the same as the one obtained by Lieb and Thirring [19, 20] for the kinetic energy only, i.e., in the absence of any interaction. We recall that this inequality of Lieb and Thirring played a crucial role in a short and elegant proof of the stability of matter [17], which had been proved earlier by Dyson and Lenard [8] by different means.

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Since our model concerns interacting particles, we cannot use the method of Lieb and Thirring to reduce the problem to the spectral analysis of a one-body operator. Instead, our strategy is closer in spirit to the work of Dyson and Lenard [8], where the Pauli principle enters only via a local exclusion principle, which implies that the local kinetic energy cannot be zero for more than one particle. A similar strategy was recently employed in [21] to study a system of anyons in two dimensions, and ideas from [21] are crucial to our proof.

2. Model and Main Result

Let $N \geq 2$, $X = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$ and let $g : \mathbb{R}^{3N} \to \mathbb{R}$ denote the function

$$g(X) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|} \,. \tag{1}$$

Our model is defined via the quadratic form

$$Q(f) = \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} g(X)^2 \, |\nabla_i f(X)|^2 \, dX \tag{2}$$

on $L^2(\mathbb{R}^{3N}, g(X)^2 dX)$, where ∇_i stands for the gradient with respect to $x_i \in \mathbb{R}^3$, and $dX = \prod_{k=1}^{N} dx_k$. The norm on the space $L^2(\mathbb{R}^{3N}, g(X)^2 dX)$ will simply be denoted by $\|\cdot\|$. The model (2) was introduced in [2], where it was shown that the quadratic form $Q(f)$ gives rise to a non-negative self-adjoint operator with purely absolutely continuous spectrum $[0, \infty)$.

We denote by $\mathcal{A}_q^N \subset L^2(\mathbb{R}^{3N}, g(X)^2 dX)$ those functions f that have the property that there is a partition of $\{1, \ldots, N\}$ into q disjoint subsets such that f is antisymmetric in the variables corresponding to each subset. In particular, $\mathcal{A}_1^N$ are the totally antisymmetric functions, while $\mathcal{A}_N^N$ are all functions in $L^2(\mathbb{R}^{3N}, g(X)^2 dX)$. Note that $\mathcal{A}_q^N \subset \mathcal{A}_{q+1}^N$. The functions $f \in \mathcal{A}_q^N$ thus represent the spatial part of totally antisymmetric functions of space and an internal degree of freedom ("spin"), where the spin is allowed to take q different values.

For given $f \in L^2(\mathbb{R}^{3N}, g(X)^2 dX)$ we define its density $\varrho_f \in L^1(\mathbb{R}^3)$ by

$$\varrho_f(x) = \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} g(\hat{X}_i)^2 |f(\hat{X}_i)|^2 d\hat{X}_i \tag{3}$$

with $\hat{X}_i = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N)$ and $d\hat{X}_i = \prod_{j \neq i} dx_j$. Our main result is the following Lieb-Thirring type inequality.

THEOREM 1. *For some constant $C > 0$ (independent of N and q) we have*

$$Q(f) \geq \frac{C}{q^{2/3}} \int_{\mathbb{R}^3} \varrho_f(x)^{5/3} dx \tag{4}$$

for all $f \in H^1(\mathbb{R}^{3N}, g(X)^2 dX) \cap \mathcal{A}_q^N$ with $\|f\| = 1$.

As mentioned in the introduction, Lieb and Thirring proved (4) in the case $g \equiv 1$. Improved bounds on the constant C in this case were obtained in [7] (see also [15, 13]).

For $\Omega \subset \mathbb{R}^3$ open and with finite measure, the ground state energy for N particles confined to Ω equals

$$E(N, q, \Omega) = \inf \left\{ Q(f) \, : \, f \in C_0^\infty(\Omega^N) \cap \mathcal{A}_q^N, \ \|f\| = 1 \right\} \tag{5}$$

for our model. An immediate corollary of (4) and Hölder's inequality is that

$$E(N, q, \Omega) \geq \frac{C}{q^{2/3}} \frac{N^{5/3}}{|\Omega|^{2/3}} , \tag{6}$$

where $|\Omega|$ denotes the volume of Ω.

Remark 1. With Δ_k denoting the Laplacian with respect to the variables x_k, we have $\sum_{k=1}^N \Delta_k \, g(X) = 0$ for all $X \in \mathbb{R}^{3N}$ with $|x_i - x_j| > 0$ for all $1 \leq i < j \leq N$. If we define

$$T_\varepsilon = \{ X \in \mathbb{R}^{3N} \, : \, |x_i - x_j| > \varepsilon \ \forall 1 \leq i < j \leq N \} \tag{7}$$

for $\varepsilon > 0$, then an integration by parts gives

$$\begin{aligned} \sum_{i=1}^N \int_{T_\varepsilon} g(X)^2 \, |\nabla_i f(X)|^2 \, dX = & \sum_{i=1}^N \int_{T_\varepsilon} |\nabla_i g(X) f(X)|^2 \, dX \\ & - 2 \int_{\partial T_\varepsilon} |f(X)|^2 g(X) \sum_{1 \leq j < k \leq N} \frac{1}{|x_j - x_k|^2} dS \, , \end{aligned} \tag{8}$$

where dS denotes the induced surface measure on ∂T_ε, the boundary of T_ε. Hence our model indeed describes particles with point interactions supported on the union of the hyperplanes defined by $x_i = x_j$ for $1 \leq i < j \leq N$.

Note that the last term in (8) is negative and vanishes in the limit $\varepsilon \to 0$ if $|f(X)|^2$ vanishes faster then linearly on the hyperplanes where $x_i = x_j$. In particular, if $f(X) = \Psi(X)/g(X)$ for $\Psi \in C_0^\infty(\mathbb{R}^{3N})$, then

$$Q(f) = \sum_{i=1}^N \int_{\mathbb{R}^{3N}} |\nabla_i \Psi(X)|^2 \, dX \, . \tag{9}$$

In general, the functions f need not vanish for coinciding arguments, however, and the energy can be lowered due to the attractive nature of the last term in (8).

Remark 2. If we replace $1/|x|$ in (1) by a smooth, strictly positive functions $\varphi(x)$ and define, accordingly, $\widetilde{g}(x) = \sum_{1 \leq i < j \leq N} \varphi(x_i - x_j)$ and $\Psi(X) = \widetilde{g}(X) f(X)$, then an integration by parts gives

$$\begin{aligned} \sum_{i=1}^N \int_{\mathbb{R}^{3N}} \widetilde{g}(X)^2 \, |\nabla_i f(X)|^2 \, dX = & \sum_{i=1}^N \int_{\mathbb{R}^{3N}} |\nabla_i \Psi(X)|^2 \, dX \\ & + \int_{\mathbb{R}^{3N}} |\Psi(X)|^2 \frac{\sum_{k=1}^N \Delta_k \widetilde{g}(X)}{\widetilde{g}(X)} dX \, . \end{aligned} \tag{10}$$

The effective potential $\widetilde{g}(X)^{-1} \sum_{k=1}^N \Delta_k \widetilde{g}(X)$ can alternatively be written as

$$2 \sum_{1 \leq i < j \leq N} \frac{\Delta \varphi(x_i - x_j)}{\widetilde{g}(X)} \tag{11}$$

and thus contains not only two-body terms but terms involving arbitrarily many particles. In particular, the presence of other particles besides i and j decreases the summands in (11) and hence weakens the interaction. This weakening leads to improved stability properties as compared to the case with only two-body interaction.

We note that our main result Theorem 1 holds for more general functions $g(X)$. As an example, one could replace $g(X)$ by

$$\sum_{1\leq i<j\leq N}\left(\frac{1}{|x_i-x_j|}-\frac{1}{a}\right) \tag{12}$$

for $a<0$, corresponding to point interactions with finite scattering length a. A Lieb-Thirring inequality holds also in this case, with a constant C that is bounded from below uniformly in a. As $a\to 0$ one obtains non-interacting particles.

Our method can also be applied to analogous models in dimensions $n>3$, where the function $1/|x|$ in (1) should be replaced by $|x|^{2-n}$, the Green's function of the Laplacian in $\mathbb{R}^n$.

The remainder of this paper is concerned with the proof of Theorem 1. The proof will be split into two main parts. In the first part in Section 3, we prove a local exclusion principle (Prop. 1) which states that the energy of n particles in a finite cube is strictly positive for $n>q$ and grows at least like $n-q$, in fact. The second part in Section 4 concerns the proof that the energy $Q(f)$ dominates the $L^2(\mathbb{R}^3)$ norm of $\nabla\sqrt{\varrho_f}$. The inequality we prove in Prop. 2 is, up to a constant, equal to a well-known inequality by the Hoffmann-Ostenhofs [12] in the case without interaction. In Section 5 we demonstrate how to obtain Theorem 1 from Propositions 1 and 2. The strategy of the proof is similar to the one in [21] where a Lieb-Thirring inequality was proved for anyons in two dimensions.

3. Local Exclusion Principle

In this section we shall prove the following.

Proposition 1. *Let $f\in H^1(\mathbb{R}^{3N},g(X)^2dX)\cap\mathcal{A}_q^N$ with $\|f\|=1$, and let $\mathcal{C}_L\subset\mathbb{R}^3$ be a cube of side length $L>0$. Then*

$$\sum_{i=1}^N\int_{\mathbb{R}^{3N}}g(X)^2\,|\nabla_i f(X)|^2\,\chi_{\mathcal{C}_L}(x_i)dX\geq\frac{k}{L^2}\left(\int_{\mathcal{C}_L}\varrho_f(x)dx-q\right) \tag{13}$$

for a constant $k>0$ independent of N, f, L and the location of the cube $\mathcal{C}_L$.

Here and in the following, χ_Q denotes the characteristic function of a set $Q\subset\mathbb{R}^3$. The constant k appearing in (13) is the same as the one in Lemma 2 below.

We note that this proposition implies that, for any integer $M\geq 1$,

$$E(N,q,\mathcal{C}_L)\geq k\frac{NM^2-qM^5}{L^2}\,. \tag{14}$$

To prove (14), simply divide the cube $\mathcal{C}_L$ into M^3 disjoint cubes with side length L/M and apply Proposition 1 for every cube. The optimal choice of M is close to $(5N/2q)^{1/3}$. In particular, the maximum over M of the right side of (14) is proportional to $kN^{5/3}/(L^2q^{2/3})$ for large N.

The proof of Proposition 1 will be divided into several lemmas. The following lemma is well-known [5]; we include its proof for the convenience of the reader.

Lemma 1. *Let Ω be an open and convex subset of $\mathbb{R}^n$ with diameter $d < \infty$, and let ω be a strictly positive function in $L^1(\Omega)$. For all $f \in H^1(\Omega, \omega(x)dx)$ with $\int_\Omega f(x)dx = 0$, we have*

$$\int_\Omega |\nabla f(x)|^2 \omega(x)dx \geq \frac{1}{2^3 3^n |\mathbb{B}^n|^2 M_\omega^3} \frac{|\Omega|^2}{d^{2(n+1)}} \int_\Omega |f(x)|^2 \omega(x)dx \,, \tag{15}$$

where $|\mathbb{B}^n|$ denotes the volume of the unit ball in $\mathbb{R}^n$ and

$$M_\omega = \sup_{r>0, x\in\Omega} \frac{1}{|\Omega \cap B_r(x)|} \frac{\int_{\Omega\cap B_r(x)} \omega(z)dz}{\inf_{z\in\Omega\cap B_r(x)} \omega(z)} \tag{16}$$

with $B_r(x)$ denoting the ball of radius r centered at x.

Proof. Since Ω is convex, we can write

$$f(x) - f(y) = \int_0^1 (x-y)\nabla f(tx + (1-t)y)dt \,. \tag{17}$$

Integrating this identity over $y \in \Omega$ and changing variables to $z = tx + (1-t)y$ gives

$$f(x) = \frac{1}{|\Omega|} \int_0^1 \frac{1}{(1-t)^{n+1}} \int_\Omega (x-z)\nabla f(z) \chi_\Omega\left(\frac{z-tx}{1-t}\right) dz\, dt \,. \tag{18}$$

Since $|x-z| = (1-t)|x-y| \leq (1-t)d$ for $x, y \in \Omega$, we obtain the bound

$$\begin{aligned} |f(x)| &\leq \frac{1}{|\Omega|} \int_\Omega |x-z||\nabla f(z)| \int_0^{1-|x-z|/d} \frac{1}{(1-t)^{n+1}}\, dt\, dz \\ &\leq \frac{d^n}{n|\Omega|} \int_\Omega \frac{1}{|x-z|^{n-1}} |\nabla f(z)| dz \,. \end{aligned} \tag{19}$$

For $x, z \in \Omega$, we have

$$\frac{1}{|x-z|^{n-1}} = (n-1)\int_0^d \frac{1}{r^n} \chi_{B_r(x)}(z) dr + \frac{1}{d^{n-1}} \,. \tag{20}$$

Hence (19) can be rewritten as

$$|f(x)| \leq \frac{d^n}{n|\Omega|} \left[(n-1)\int_0^d \frac{1}{r^n} \int_{\Omega\cap B_r(x)} |\nabla f(z)| dz\, dr + \frac{1}{d^{n-1}} \int_\Omega |\nabla f(z)| dz \right] . \tag{21}$$

We introduce the maximal function

$$m_\omega(x) = \sup_{r>0} \frac{\int_{\Omega\cap B_r(x)} |\nabla f(z)| \omega(z) dz}{\int_{\Omega\cap B_r(x)} \omega(z) dz} \,. \tag{22}$$

Then

$$\int_{\Omega\cap B_r(x)} |\nabla f(z)| dz \leq |\mathbb{B}^n| r^n M_\omega m_\omega(x) \tag{23}$$

with M_ω defined in (16). In particular, from (21) we obtain the bound

$$|f(x)| \leq \frac{d^{n+1}}{|\Omega|} |\mathbb{B}^n| M_\omega m_\omega(x) \,, \tag{24}$$

and thus

$$\int_\Omega |f(x)|^2\omega(x)dx \leq \frac{d^{2(n+1)}}{|\Omega|^2}|\mathbb{B}^n|^2 M_\omega^2 \int_\Omega m_\omega(x)^2\omega(x)dx\,. \tag{25}$$

We further define

$$N_\omega = \sup_{r>0, x\in\Omega} \frac{\int_{\Omega\cap B_{3r}(x)}\omega(z)dz}{\int_{\Omega\cap B_r(x)}\omega(z)dz} \tag{26}$$

and note that $N_\omega \leq 3^n M_\omega$. Proceeding as in the proof of Theorem 1(c) in Sect. I.3.1 of [22] we see that

$$\int_\Omega m_\omega(x)^2\omega(x)dx \leq 2^3 N_\omega \int_\Omega |\nabla f(x)|^2\omega(x)dx\,. \tag{27}$$

In combination with (25) this proves our claim. □

Let now $\mathcal{C} = (0,1)^3 \subset \mathbb{R}^3$ denote the unit cube in $\mathbb{R}^3$.

Lemma 2. *Assume that $f \in H^1(\mathbb{R}^3\times\mathbb{R}^3)$ is antisymmetric, i.e., $f(x_1,x_2) = -f(x_2,x_1)$ for $x_1,x_2\in\mathbb{R}^3$. For almost every $(x_3,\ldots,x_N)\in\mathbb{R}^{3(N-2)}$ we have*

$$\sum_{i=1}^2 \int_{\mathcal{C}^2} g(X)^2\,|\nabla_i f(x_1,x_2)|^2\,dx_1dx_2 \geq 2k\int_{\mathcal{C}^2} g(X)^2|f(x_1,x_2)|^2 dx_1dx_2 \tag{28}$$

for a constant $k>0$ that does not depend on N or the variables $(x_3,\ldots,x_N)$.

Proof. Since f is antisymmetric, it has zero average with respect to the variables (x_1,x_2). Hence we can apply the previous lemma, with $\Omega = \mathcal{C}\times\mathcal{C}\subset\mathbb{R}^6$ and $\omega(x_1,x_2) = g(X)^2$. The claim is thus proved if we can show that the corresponding M_ω in (16) is bounded independently of N and $(x_3,\ldots,x_N)$.

Consider a ball $B\subset\mathbb{R}^6$ of radius r centered at some point $(w_1,w_2)\in\mathcal{C}\times\mathcal{C}$. Without loss of generality, we may assume that $r\leq\sqrt{6}$, for otherwise $\mathcal{C}\times\mathcal{C}\subset B$ and hence the expression after the sup in (16) is independent of r, and is thus the same as for $r=\sqrt{6}$. We can get a lower bound on $g(X)$ by taking the minimum in each term in the sum in (1). This gives

$$g(X) \geq \frac{1}{|w_1-w_2|+2r} + \sum_{j\geq 3}\left(\frac{1}{|w_1-x_j|+r}+\frac{1}{|w_2-x_j|+r}\right) + \sum_{3\leq j<l\leq N}\frac{1}{|x_j-x_l|} \tag{29}$$

for $(x_1,x_2)\in B$. On the other hand, simple estimates yield

$$\int_B \frac{1}{|x_1-x_2|^2}dx_1dx_2 \leq \text{const}\,\frac{r^6}{(|w_1-w_2|+2r)^2} \tag{30}$$

and

$$\int_B \frac{1}{|x_1-x_j||x_1-x_l|}dx_1dx_2 \leq \text{const}\,\frac{r^6}{(|w_1-x_j|+2r)(|w_1-x_l|+2r)} \tag{31}$$

for $j,l\geq 3$, etc. These imply, in particular, that

$$\int_B g(X)^2 dx_1dx_2 \leq \text{const}\, r^6 \inf_{(x_1,x_2)\in B} g(X)\,. \tag{32}$$

Since Ω is a cube in $\mathbb{R}^6$, we also have $|\Omega\cap B|\geq 2^{-6}r^6$ for $r\leq 1$, and hence M_ω is bounded independently of N and $(x_3,\ldots,x_N)$. □

Lemma 3. *Let* $A \subset \{1, \ldots, N\}$, $X_A = \{x_i\}_{i \in A}$ *and* $dX_A = \prod_{l \in A} dx_l$. *For* $f \in H^1(\mathbb{R}^{3N}, g(X)^2 dX) \cap \mathcal{A}_q^N$ *we have, for almost every* $X \setminus X_A$,

$$\sum_{i \in A} \int_{\mathcal{C}^{|A|}} g(X)^2 \left|\nabla_i f(X)\right|^2 dX_A \geq k\,(|A| - q) \int_{\mathcal{C}^{|A|}} g(X)^2 |f(X)|^2 dX_A \qquad (33)$$

with k *as in Lemma 2.*

Proof. The set A can be divided into q subsets B_j (some of which may be empty) such that f is antisymmetric in the variables in each subset. We may assume that $|A| > q$, in which case at least one such subset B_j contains more than one element. We shall show that for all such B_j

$$\sum_{i \in B_j} \int_{\mathcal{C}^{|B_j|}} g(X)^2 \left|\nabla_i f(X)\right|^2 dX_{B_j} \geq k\,(|B_j| - 1) \int_{\mathcal{C}^{|B_j|}} g(X)^2 |f(X)|^2 dX_{B_j} \qquad (34)$$

for each fixed $X \setminus X_{B_j}$. Summing over j this implies the result.

Inequality (34) follows immediately from Lemma 2, noting that the sum over $|B_j| = n \geq 2$ elements can be written as $1/[2(n-1)]$ times a sum over all ordered pairs, yielding the inequality with $|B_j| - 1$ replaced by $|B_j|$. □

Since $g(X)$ is invariant under translation and rotation of all the coordinates x_i, inequality (33) clearly holds for all unit cubes in $\mathbb{R}^3$, irrespective of their location or orientation. A simple scaling argument shows that it actually holds for all cubes of side length $L > 0$ if the right side is divided by L^2.

Proof of Proposition 1. Using that

$$1 = \sum_{A \subset \{1,\ldots,N\}} \prod_{l \in A} \chi_{\mathcal{C}_L}(x_l) \prod_{l \notin A} \left(1 - \chi_{\mathcal{C}_L}(x_l)\right) \qquad (35)$$

we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} g(X)^2 \left|\nabla_i f(X)\right|^2 \chi_{\mathcal{C}_L}(x_i) dX \\ &\quad = \sum_{A \subset \{1,\ldots,N\}} \sum_{i \in A} \int_{\mathbb{R}^{3N}} g(X)^2 \left|\nabla_i f(X)\right|^2 \prod_{l \in A} \chi_{\mathcal{C}_L}(x_l) \prod_{l \notin A} \left(1 - \chi_{\mathcal{C}_L}(x_l)\right) dX\,. \qquad (36) \end{aligned}$$

Applying the previous lemma (and the remark after its proof), we obtain the lower bound

$$\frac{k}{L^2} \sum_{A \subset \{1,\ldots,N\}} (|A| - q) \int_{\mathbb{R}^{3N}} g(X)^2 \left|f(X)\right|^2 \prod_{l \in A} \chi_{\mathcal{C}_L}(x_l) \prod_{l \notin A} \left(1 - \chi_{\mathcal{C}_L}(x_l)\right) dX\,, \qquad (37)$$

which is equal to the right side of (13). □

4. Density Bound

Recall that the one-particle density ϱ_f of a function $f \in L^2(\mathbb{R}^{3N}, g(X)^2 dX)$ is defined in (3).

Proposition 2. *Let $f \in H^1(\mathbb{R}^{3N}, g(X)^2 dX)$. Then its one-particle density ϱ_f satisfies $\sqrt{\varrho_f} \in H^1(\mathbb{R}^3, dx)$ and*

$$Q(f) \geq \frac{1}{9} \int_{\mathbb{R}^3} \left| \nabla \sqrt{\varrho_f(x)} \right|^2 dx \,. \tag{38}$$

It is well known that in case $g \equiv 1$, (38) holds with the prefactor $\frac{1}{9}$ replaced by 1 [12].

Proof. We first show that it is enough to prove the inequality for symmetric functions f. For any f, define $\tilde{f}$ by

$$\tilde{f}(X) = \left(\frac{1}{N!} \sum_{\pi \in S_N} |f(x_{\pi(1)}, \dots, x_{\pi(N)})|^2 \right)^{1/2} , \tag{39}$$

where S_N denotes the permutation group. The function $\tilde{f}$ is obviously symmetric, and $\varrho_f = \varrho_{\tilde{f}}$. It is easy to see that the map $|f|^2 \mapsto Q(|f|)$ is convex (see, e.g., Lemma A.1 in [18]), hence

$$Q(\tilde{f}) \leq Q(|f|) \leq Q(f) \,. \tag{40}$$

In particular, it is enough to prove (38) for symmetric f.

Assume now that f is symmetric. Then the density ϱ_f can alternatively be written as

$$\begin{aligned}
\varrho_f(x_1) &= N \int_{\mathbb{R}^{3(N-1)}} g(X)^2 |f(X)|^2 dx_2 \cdots dx_N \\
&= N(N-1) \int_{\mathbb{R}^{3(N-1)}} \frac{1}{|x_1 - x_2|^2} |f(X)|^2 dx_2 \cdots dx_N \\
&\quad + N(N-1)(N-2) \int_{\mathbb{R}^{3(N-1)}} \frac{1}{|x_1 - x_2|} \frac{1}{|x_1 - x_3|} |f(X)|^2 dx_2 \cdots dx_N \\
&\quad + 2N(N-1)(N-2) \int_{\mathbb{R}^{3(N-1)}} \frac{1}{|x_1 - x_2|} \frac{1}{|x_2 - x_3|} |f(X)|^2 dx_2 \cdots dx_N \\
&\quad + N(N-1)(N-2)(N-3) \int_{\mathbb{R}^{3(N-1)}} \frac{1}{|x_1 - x_2|} \frac{1}{|x_3 - x_4|} |f(X)|^2 dx_2 \cdots dx_N \\
&\quad + N \int_{\mathbb{R}^{3(N-1)}} R(x_2, \dots, x_N) |f(X)|^2 dx_2 \cdots dx_N
\end{aligned} \tag{41}$$

where R denotes the sum of all the terms in g not depending on x_1. Let us denote the various integrals on the right side by $\varrho_1, \dots, \varrho_5$, and the prefactors in front of the integrals by $n_1, \dots, n_5$. That is, $\varrho_f = \sum_{j=1}^5 n_j \varrho_j$.

Since $1/|x_1 - x_2|$ is invariant under translations of both x_1 and x_2, we have

$$(\nabla_1 + \nabla_2) \frac{1}{|x_1 - x_2|^2} |f(X)|^2 = 2 \operatorname{Re} \frac{1}{|x_1 - x_2|^2} f(X)^* (\nabla_1 + \nabla_2) f(X) \,. \tag{42}$$

Integrating this identity over $x_2, \dots, x_N$, we obtain

$$\nabla_1 \varrho_1(x_1) = 2 \operatorname{Re} \int \frac{1}{|x_1 - x_2|^2} f(X)^* (\nabla_1 + \nabla_2) f(X) dx_2 \cdots dx_N \,. \tag{43}$$

The Schwarz inequality then implies that

$$|\nabla_1 \varrho_1(x_1)|^2 \leq 4 \varrho_1(x_1) \int \frac{1}{|x_1 - x_2|^2} |(\nabla_1 + \nabla_2) f(X)|^2 dx_2 \cdots dx_N \,. \tag{44}$$

In particular,

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho_1}|^2 = \frac{1}{4}\int_{\mathbb{R}^3} \frac{1}{\varrho_1(x)}\,|\nabla\varrho_1(x)|^2\,dx \\ \leq \int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|^2}|(\nabla_1+\nabla_2)f|^2 dX \leq 4\int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|^2}|\nabla_1 f|^2 dX\,, \quad (45)$$

where we have again used the symmetry of f in the last step.

In the same way one proceeds for the remaining parts of the density. The result is that

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho_2}|^2 \leq \int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_1-x_3|}|(\nabla_1+\nabla_2+\nabla_3)f|^2 dX \\ \leq 3\int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_1-x_3|}|\nabla_1 f|^2 dX \\ + 6\int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_1-x_3|}|\nabla_2 f|^2 dX\,, \quad (46)$$

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho_3}|^2 \leq \int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_2-x_3|}|(\nabla_1+\nabla_2+\nabla_3)f|^2 dX \\ \leq 6\int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_2-x_3|}|\nabla_1 f|^2 dX \\ + 3\int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_2-x_3|}|\nabla_2 f|^2 dX\,, \quad (47)$$

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho_4}|^2 \leq \int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_3-x_4|}|(\nabla_1+\nabla_2)f|^2 dX \\ \leq 4\int_{\mathbb{R}^{3N}} \frac{1}{|x_1-x_2|}\frac{1}{|x_3-x_4|}|\nabla_1 f|^2 dX \quad (48)$$

and

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho_5}|^2 \leq \int_{\mathbb{R}^{3N}} R(x_2,\ldots,x_N)|\nabla_1 f|^2 dX\,. \quad (49)$$

Summing up, using again the symmetry of f and convexity of the map $\varrho \mapsto \int |\nabla\sqrt{\varrho}|^2$, this gives

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho}|^2 \leq \sum_{j=1}^{5} n_j \int_{\mathbb{R}^3} |\nabla\sqrt{\varrho_j}|^2 \leq 9N\int_{\mathbb{R}^{3N}} g(X)^2|\nabla_1 f|^2 dX = 9Q(f)\,. \quad (50)$$

This completes the proof of our claim. □

5. Proof of Theorem 1

We will deduce Theorem 1 from Propositions 1 and 2, following a similar strategy as in [21].

First, we note that it is enough to prove the theorem for $N > 2q$. If $N \leq 2q$, the statement can be easily deduced from Proposition 2 alone. In fact, applying the Sobolev inequality $\|\nabla\varphi\|_2^2 \geq S\|\varphi\|_6^2$ and Hölder's inequality, we have

$$Q(f) \geq \frac{1}{9}\|\nabla\sqrt{\varrho_f}\|_2^2 \geq \frac{S}{9}\|\varrho_f\|_3 \geq \frac{S}{9}\|\varrho_f\|_{5/3}^{5/3}\|\varrho_f\|_1^{-2/3} \geq \frac{S}{9}\frac{1}{(2q)^{2/3}}\int_{\mathbb{R}^3} \varrho_f(x)^{5/3}dx \quad (51)$$

in this case.

Assume now that $N > 2q$. For given $f \in H^1(\mathbb{R}^{3N}, g(X)^2dX) \cap \mathcal{A}_q^N$ and $\varepsilon > 0$, we can choose a cube $Q_0 \subset \mathbb{R}^3$ such that

$$\int_{Q_0} \varrho_f(x)dx \geq 2q \quad \text{and} \quad \int_{Q_0} \varrho_f(x)^{5/3}dx \geq (1-\varepsilon)\int_{\mathbb{R}^3} \varrho_f(x)^{5/3}dx\,. \tag{52}$$

Suppose that Q_0 is divided into finitely many disjoint cubes Q_i. With the aid of Proposition 1, we can bound

$$Q(f) \geq \sum_i \sum_{j=1}^N \int_{\mathbb{R}^{3N}} g(X)^2 \left|\nabla_j f(X)\right|^2 \chi_{Q_i}(x_j)dX \geq \sum_i \frac{k}{|Q_i|^{2/3}} \left[\int_{Q_i} \varrho_f(x)dx - q\right]_+ , \tag{53}$$

where $[t]_+ = \max\{t, 0\}$ denotes the positive part of a number $t \in \mathbb{R}$. Note that each summand on the left side is obviously non-negative, hence we can use the positive part on the right side. Moreover, from Proposition 2, we obtain the bound

$$Q(f) \geq \sum_i \frac{1}{9} \int_{Q_i} \left|\nabla\sqrt{\varrho_f(x)}\right|^2 dx\,. \tag{54}$$

In combination, (53) and (54) imply that

$$Q(f) \geq \sum_i \left(\frac{\lambda k}{|Q_i|^{2/3}} \left[\int_{Q_i} \varrho_f(x)dx - q\right]_+ + \frac{1-\lambda}{9} \int_{Q_i} \left|\nabla\sqrt{\varrho_f(x)}\right|^2 dx \right) \tag{55}$$

for any $0 \leq \lambda \leq 1$.

To construct the division of Q_0, we proceed as follows [21]. Divide the cube Q_0 into 8 disjoint cubes of half the size. If the integral of ϱ_f over one of the subcubes is less than $2q$, the subcube will not be divided further and will be marked A. If all subcubes are marked A, then the division is undone and the cube Q_0 is marked B. For all the subcubes with integral of ϱ_f bigger or equal to $2q$, we iterate this procedure. At the end, Q_0 is thus covered by finitely many disjoint subcubes of type either A or B.

On every subcube Q_i marked B, we have $2q \leq \int_{Q_i} \varrho_f(x)dx < 16q$. This implies, in particular, that

$$\left[\int_{Q_i} \varrho_f(x)dx - q\right]_+ \geq \frac{1}{2} \int_{Q_i} \varrho_f(x)dx\,. \tag{56}$$

The Poincaré-Sobolev inequality on a cube [16, Thm. 8.12] yields

$$\int_{Q_i} \left|\nabla\sqrt{\varrho_f(x)}\right|^2 dx \geq \tilde{S} \left\| \sqrt{\varrho_f} - |Q_i|^{-1} \int_{Q_i} \sqrt{\varrho_f} \right\|_{L^6(Q_i)}^2 \tag{57}$$

for some constant $\widetilde{S} > 0$ independent of the size or location of Q_i. The triangle inequality in $L^6(Q_i)$ and a simple Schwarz inequality further imply that

$$\left\| \sqrt{\varrho_f} - |Q_i|^{-1} \int_{Q_i} \sqrt{\varrho_f} \right\|_{L^6(Q_i)}^2 \geq \frac{1}{2}\|\varrho_f\|_3 - |Q_i|^{-2/3} \int_{Q_i} \varrho_f\,. \tag{58}$$

An application of Hölder's inequality finally gives

$$\int_{Q_i} \left|\nabla\sqrt{\varrho_f(x)}\right|^2 dx \geq \frac{\tilde{S}}{2} \frac{\int_{Q_i} \varrho_f^{5/3}}{\left(\int_{Q_i} \varrho_f\right)^{2/3}} - \tilde{S}|Q_i|^{-2/3} \int_{Q_i} \varrho_f\,. \tag{59}$$

If we choose $0 < \lambda < 1$ in such a way that

$$\frac{k\lambda}{2} = \frac{(1-\lambda)\tilde{S}}{9}\,, \tag{60}$$

i.e., $\lambda = (1+9k/(2\tilde{S}))^{-1}$, we conclude that the contribution of a B cube to the energy in (55) is bigger or equal to

$$\frac{2^{-11/3}}{\frac{2}{k}+\frac{9}{\tilde{S}}}\frac{1}{q^{2/3}}\int_{Q_i}\varrho_f(x)^{5/3}dx\,. \tag{61}$$

Consider now a cube labeled A, where $\int_{Q_i}\varrho_f < 2q$. Pick a $\kappa > 2$ and assume, for the moment, that

$$\int_{Q_i}\varrho_f^{5/3} > \kappa|Q_i|^{-2/3}\left(\int_{Q_i}\varrho_f\right)^{5/3}\,. \tag{62}$$

In this case, it follows from (59) that

$$\int_{Q_i}\left|\nabla\sqrt{\varrho_f(x)}\right|^2 dx \geq \frac{\tilde{S}}{2\kappa}\,(\kappa-2)\,\frac{\int_{Q_i}\varrho_f^{5/3}}{(2q)^{2/3}}\,. \tag{63}$$

Hence, for our choice of λ, the contribution from such an A cube to the energy in (55) is at least

$$\left(1-\frac{2}{\kappa}\right)\frac{2^{-5/3}}{\frac{2}{k}+\frac{9}{\tilde{S}}}\frac{1}{q^{2/3}}\int_{Q_i}\varrho_f(x)^{5/3}dx\,. \tag{64}$$

We are left with studying those A cubes where

$$\int_{Q_i}\varrho_f^{5/3} \leq \kappa|Q_i|^{-2/3}\left(\int_{Q_i}\varrho_f\right)^{5/3}\,. \tag{65}$$

We shall show that their contribution to the total integral of $\varrho_f^{5/3}$ is dominated by the contribution of all the B cubes. In order to see this, we note that our decomposition of Q_0 into subcubes can be organized in a tree. (Compare with Fig. 3 in [21].) Every A cube can be associated with a B cube, namely if it can be found by going back in the tree from the B cube, possibly all the way to Q_0, and then one step forward. This allows to divide the A cubes into groups labeled by the B cubes. This division is not unique, in general, but this is not important; the only thing that matters is that every A cube can be associated with a B cube in this way. Pick a B cube (call it Q_B) at level $l \in \mathbb{N}$ of the tree, and let $\mathcal{A}(Q_B)$ be the set of all those associated A cubes that satisfy (65). Since at every level $1 \leq j \leq l$ of the tree there are at most 7 A cubes in $\mathcal{A}(Q_B)$, we can bound

$$\sum_{Q_i\in\mathcal{A}(Q_B)}\int_{Q_i}\varrho_f^{5/3} \leq \frac{7\kappa(2q)^{5/3}}{|Q_0|^{2/3}}\sum_{j=1}^{l}4^j \leq \frac{7\kappa(2q)^{5/3}}{3|Q_0|^{2/3}}4^{l+1}\,. \tag{66}$$

On the other hand, for the associated B cube Q_B we have

$$\int_{Q_B}\varrho_f^{5/3} \geq \frac{\left(\int_{Q_b}\varrho_f\right)^{5/3}}{|Q_B|^{2/3}} \geq \frac{(2q)^{5/3}}{|Q_B|^{2/3}} = \frac{(2q)^{5/3}}{|Q_0|^{2/3}}4^l\,, \tag{67}$$

and thus

$$\sum_{Q_i \in \mathcal{A}(Q_B)} \int_{Q_i} \varrho_f^{5/3} \leq \frac{28}{3} \kappa \int_{Q_B} \varrho_f^{5/3} . \tag{68}$$

In particular, the contribution to $\int \varrho_f^{5/3}$ of all the A cubes satisfying (65) is bounded by $28\kappa/3$ times the contribution of all the B cubes. In other words, the contribution to $\int \varrho_f^{5/3}$ of all the B cubes is bounded from below by $(1+28\kappa/3)^{-1}$ times the contribution of *both* the B cubes and the A cubes satisfying (65).

After summing over all cubes, we thus get the lower bound

$$Q(f) \geq \frac{\tilde{C}}{q^{2/3}} \int_{Q_0} \varrho_f(x)^{5/3} dx \geq \frac{\tilde{C}}{q^{2/3}} (1-\varepsilon) \int_{\mathbb{R}^3} \varrho_f(x)^{5/3} dx \,, \tag{69}$$

with

$$\tilde{C} = \frac{2^{-11/3}}{\frac{2}{k} + \frac{9}{\tilde{S}}} \sup_{\kappa > 2} \min \left\{ \frac{1}{1 + \frac{28}{3}\kappa} \,,\, 4\left(1 - \frac{2}{\kappa}\right) \right\} = \frac{2^{-2/3}}{\frac{2}{k} + \frac{9}{\tilde{S}}} \frac{239 - \sqrt{56977}}{48} > 0 \,. \tag{70}$$

Since $\varepsilon > 0$ was arbitrary, this proves (4) with a constant

$$C = \min \left\{ \tilde{C} \,,\, \frac{2^{-2/3} S}{9} \right\} . \tag{71}$$

$\square$

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